Vanishing theorems and Orlik–Solomon algebras

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Problem definition

Let k be a field. Put $E=\{0,\dots,n\}$. An (essential central) **hyperplane arrangement** over k is a r-dimensional subspace $L\subseteq k^E$ not contained in any coordinate hyperplane of k^E. Consider the embedding $\mathbb{P}L\subset\mathbb{P}^n$ with a fixed system of coordinates $x_i, i\in E$. As a toric variety, $\mathbb{P}^n$ contains the open torus $\mathbb{P}T=\mathbb{G}_m^E/\mathbb{G}_m$. We set $\mathbb{P}L^\circ=\mathbb{P}L\cap\mathbb{P}T$ to be the arrangement complement.

Definition 1. The **Orlik–Solomon algebra** $\mathrm{OS}(L)$ is the graded-commutative ring

$$\mathrm{OS}(L):=\frac{\bigwedge^\bullet\mathbb{Z}^E}{\langle\partial e_S : S\subseteq E\text{ dependent}\rangle},$$

where $e_S=e_{s_1}\wedge\cdots\wedge e_{s_p}$ for an ordered subset $S=\{s_1,\dots,s_p\}$ and $\partial e_S=\sum_{q=0}^{p-1}(-1)^q e_{S\setminus s_{p-q}}$ is the Koszul differential. The **reduced Orlik–Solomon algebra** is defined to be $\overline{\mathrm{OS}}(L):=\ker\partial$.

We have the classical result:

Theorem 2 (Arnol'd–Brieskorn). When $k=\mathbb{C}$ and L is a hyperplane arrangement, then we have an algebra isomorphism

$$\overline{\mathrm{OS}}(L)_{\mathbb{C}}\cong H^\bullet_{\mathrm{Betti}}(\mathbb{P}L^\circ,\mathbb{C}).$$

Question 3. Does the reduced Orlik–Solomon algebra for an arrangement over an *arbitrary field* k afford a geometric interpretation?

Wonderful varieties and tautological bundles

- The **permutohedral variety** is the toric variety X_E associated to the fan given by the normal fan of the permutohedron:

$$\mathrm{Perm}_n=\mathrm{conv}\{(\sigma(0),\sigma(1),\dots,\sigma(n)):\sigma\in\mathrm{Aut}(E)\}\subset\mathbb{R}^E.$$

  The open torus is $\mathbb{P}T=\mathbb{G}_m^E/\mathbb{G}_m$.
- The **de Concini–Procesi wonderful compactification** is the closure of $\mathbb{P}L^\circ=\mathbb{P}L\cap\mathbb{P}T$ in X_E. The boundary $D_L:=W_L\setminus\mathbb{P}L^\circ$ is a simple normal crossings divisor.
- The **Hodge–de Rham spectral sequence**

$$E_1^{p,q}=H^q(W_L,\Omega^p_{W_L}(\log D_L))\Longrightarrow\mathbb{H}^{p+q}(W_L,\Omega^\bullet_{W_L}(\log D_L)).$$

- Let $T=\mathbb{G}_m^E$ act on k^E by the dual of the standard representation: $t\cdot(v_i)_{i\in E}=(t_i^{-1}v_i)_{i\in E}$. Berget–Eur–Spink–Tseng attach to $L\subset k^E$ two T-equivariant vector bundles $\mathcal{S}_L$ and $\mathcal{Q}_L$ on X_E

$$\mathcal{S}_L:=\text{the subbundle of } k^E\otimes\mathcal{O}_{X_E}\text{ whose fiber at } 1_{\mathbb{P}T}\text{ is } L,$$
$$\mathcal{Q}_L:=\text{the quotient } (k^E\otimes\mathcal{O}_{X_E})/\mathcal{S}_L.$$

  These are the **tautological sub- and quotient bundles** on X_E associated to L.
- **Key connection.** (i) The space W_L is the vanishing locus of $(1,\dots,1)\in k^E$, understood as a section of $\mathcal{Q}_L$; (ii) there is an Euler-type exact sequence on W_L:

$$0\to\Omega^1_{W_L}(\log D_L)\to\mathcal{S}_L^\vee|_{W_L}\to\mathcal{O}_{W_L}\to 0.$$

Main results

Theorem A. For *any field* k and any linear subspace $L\subseteq k^E$,

$$H^i(X_E,\ \bigwedge^p\mathcal{S}_L\otimes\bigwedge^q\mathcal{Q}_L)=0\quad\text{for all } i>0,\ p,q\geq 0.$$

Corollary B. Let k be *any field* and $L\subseteq k^E$ a hyperplane arrangement. Then

1. The log Hodge cohomology vanishing, and algebra isomorphisms:

$$H^q(W_L,\ \Omega^p_{W_L}(\log D_L))=0\quad\text{for all } q>0, p\geq 0.$$
$$\overline{\mathrm{OS}}(L)\xrightarrow{\sim}H^0(W_L,\ \Omega^\bullet_{W_L}(\log D_L))\xrightarrow{\sim}\mathbb{H}^\bullet(W_L,\ \Omega^\bullet_{W_L}(\log D_L)).$$

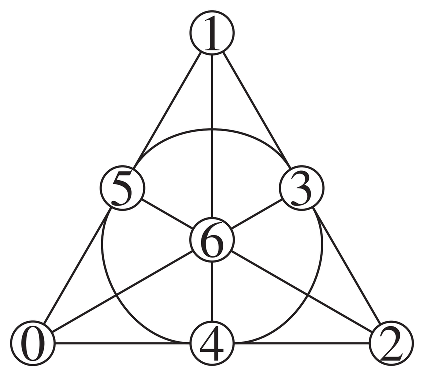
   [*Slogan:* Orlik–Solomon algebras are realised as **logarithmic de Rham cohomology** of (W_L,D_L).]
2. Let $T_L(x,y)$ be the Tutte polynomial of L. Set $h_L(u,v)=\sum_{p,q}h^0(\bigwedge^p\mathcal{S}_L\otimes\bigwedge^q\mathcal{Q}_L)u^pv^q$. We have

$$h_L(u,v)=v^{\#E-r}T_{\mathrm{M}(L)}(u+1,v^{-1}+1)=\sum_{A\subseteq E}u^{r-\mathrm{rk}_{\mathrm{M}}(A)}v^{\#E\setminus A-(r-\mathrm{rk}_{\mathrm{M}}(A))}.$$

Over $k=\mathbb{C}$, Item 1 recovers Arnol'd–Brieskorn's isomorphism.

Example: Fano arrangement

- Work over $k=\mathbb{F}_2$ and let $Z=\mathbb{P}^2(\mathbb{F}_2)$ (7 points). The 7 $\mathbb{F}_2$-lines in $\mathbb{P}^2$ form the **Fano arrangement**, with matroid $\mathrm{PG}(2,2)$.



  Incidence graph: lines in $\check{\mathbb{P}}^2(\mathbb{F}_2)$ (nodes) with intersections (edges).
- The wonderful compactification is the blow-up $p_1: W\to\mathbb{P}^2$ at Z. Write α for the pullback of a hyperplane and E for the exceptional divisor. The boundary is

$$D=p_1^{-1}(\Delta)+E,$$

  where Δ is the union of the 7 arrangement lines.
- One has $H^i(W,\Omega^p_W(\log D))=0$ for all $i>0$ and $p=0,1,2$. Therefore the Hodge–de Rham spectral sequence degenerates and

$$H^0(\Omega^\bullet_W(\log D))\ \cong\ \mathbb{H}^\bullet(\Omega^\bullet_W(\log D)).$$

- The characteristic polynomial of $\mathrm{PG}(2,2)$ is

$$\chi(t)=(t-1)(t-2)(t-4)=(t-1)(t^2-6t+8),$$

  so the reduced characteristic polynomial is $\bar{\chi}(t)=\chi(t)/(t-1)=t^2-6t+8$. Its coefficients give the graded dimensions of $\overline{\mathrm{OS}}(\mathrm{PG}(2,2))$, hence

$$h^0(\Omega^1_W(\log D))=6,\qquad h^0(\Omega^2_W(\log D))=8.$$

- Fix an ordering (here $0<1<\cdots<6$). A **broken circuit** is $C\setminus\{\min C\}$ for a circuit C, and an **nbc-set** is a subset containing no broken circuit. Then OS has an nbc basis given by the classes of e_S with S nbc. For $\overline{\mathrm{OS}}=\ker\partial$, a convenient basis is given by the **reduced nbc monomials** $\partial(e_0\wedge e_S)$ with S nbc and $0\notin S$. Under the ordering $0<1<\cdots<6$, the reduced nbc index sets can be listed as

$$0\underline{1},0\underline{2},0\underline{3},0\underline{4},0\underline{5},0\underline{6},\qquad 0\underline{12},0\underline{13},0\underline{14},0\underline{16},0\underline{25},0\underline{26},0\underline{34},0\underline{35}.$$

  The underlined subsets are exactly those indexing the reduced nbc basis elements.
- Concretely, $H^0(W,\Omega^1_W(\log D))$ is generated by

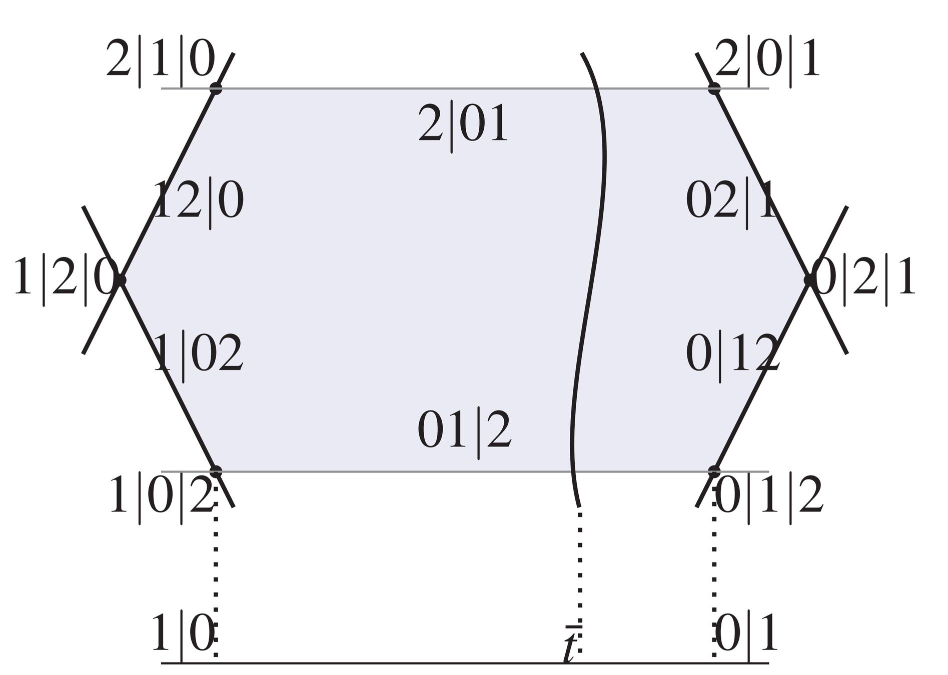
$$d\log\frac{x_1}{x_0},\ d\log\frac{x_2}{x_0},\ \dots,\ d\log\frac{x_6}{x_0},$$

  and $H^0(W,\Omega^2_W(\log D))$ is generated by wedges

$$d\log\frac{x_i}{x_0}\wedge d\log\frac{x_j}{x_0}\quad(i,j)\in\{(1,2),(1,3),(1,4),(1,6),(2,5),(2,6),(3,4),(3,5)\},$$

  where $d\log(x_i/x_j)=d\log x_i-d\log x_j$.

Proof ingredient: a deletion map



Definition 4. The projection $\mathbb{G}_m^E\to\mathbb{G}_m^{E\setminus\{n\}}$ extends to a flat projective toric morphism (the **deletion map**) $f: X_E\to X_{E\setminus\{n\}}$. Its fibres are chains of $\mathbb{P}^1$'s indexed by ordered set partitions.

Schematic (left): $f: X_{0,1,2}\to X_{0,1}$, strata labelled by ordered set partitions.

Set $\mathcal{E}_L=\mathcal{S}_L\oplus\mathcal{Q}_L$. Induct on $\#E$.

- The restrictions of $\bigwedge^d\mathcal{E}_L$ to each component of the fibre are direct sums of line bundles; splitting types determined by Eur.
- Relate the cohomology of $\bigwedge^d\mathcal{E}_L$ to those of derived pushforwards $R^if_*(\bigwedge^d\mathcal{E}_L)$ via the Leray spectral sequence, and then reduce to $f_*\bigwedge^d$ by the theorem of cohomology and base change.

References

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