The q-Klyachko algebra and geometry

Ruizhen Liu

University of Toronto

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For a sequence $I = (a_1, \ldots, a_n)$, write

$$u^I = \prod_{1 \le i \le n} u_i^{a_i}.$$

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$$2u_2^2 = u_1u_2 + u_2u_3,\ \ 2u_1^2 = u_1u_2,\ \ 2u_3^2 = u_2u_3.$$

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Question

Given a sequence $I = (a_1, \ldots, a_n)$ with $|I| = n$, do we have the log-concavity inequality

$$(\deg u^I)^2 \geq \deg u^{I-e_i+e_{i+1}} \cdot \deg u^{I-e_{i+1}+e_i}$$

for all $i = 1, \ldots, n$?

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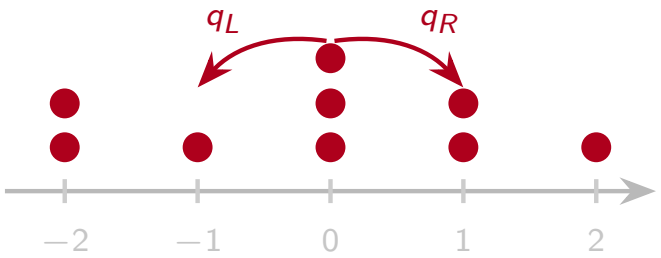
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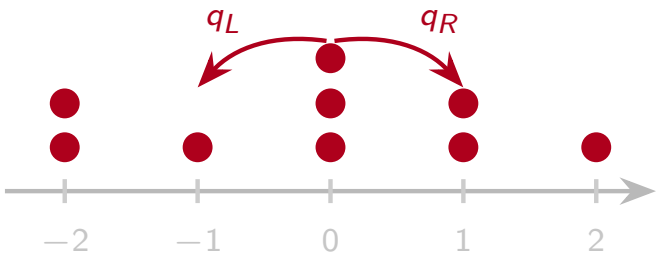
$$\eta \mapsto \eta - \delta_i + \delta_{i-1} \quad \text{(prob. } q_L\text{)}, \qquad \eta \mapsto \eta - \delta_i + \delta_{i+1} \quad \text{(prob. } q_R\text{)}.$$



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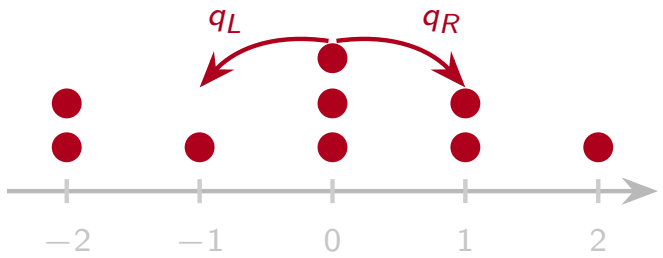
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The probability of each final state is independent of the order in which unstable sites are chosen.

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Proposition (Nadeau–Tewari)

If η is supported on $[n]$ and $|\eta| = n$, let $I = (\eta(1), \ldots, \eta(n))$. Then

$$\text{Prob}\left(\begin{array}{c}\text{final state has one particle}\\ \text{at each point of } [n]\end{array}\right) = \frac{1}{(n)_q!}\deg_{n,q} u^I.$$

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Consider the change of variables

$$\phi\colon \mathbf{R}[z_1, \ldots, z_{n+1}] \longrightarrow \mathbf{R}[u_1, \ldots, u_n], \qquad z_i \longmapsto u_i - u_{i-1},$$

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Then

$$\deg_{n,q}\big(\phi(F)\big) = 0$$

for every homogeneous $F \in \mathrm{QSym}^+_{n+1}$ of degree n.

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- $q = 1$: a simplicial fan model and Hodge theory for simplicial polytopes.
- $q > 0$: deform both pictures.

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Hypersimplices:

$$\Delta(i, n+1) = \left\{ x \in [0,1]^{n+1} : x_1 + \cdots + x_{n+1} = i \right\}, \qquad 1 \leq i \leq n.$$

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Proposition (Postnikov; Berget–Spink–Tseng; Nadeau–Tewari)

For a sequence $I = (a_1, \ldots, a_n)$ with $|I| = n$,

$$\deg u^I = [x^I] \operatorname{Vol}\left(\sum_{i=1}^{n} x_i \Delta(i, n+1) \right).$$

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Log-concavity of $\deg u^I$ follows from the Alexandrov–Fenchel inequality.

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- Hodge index inequality gives log-concavity of $\deg u^I$.
- Also, $\mathrm{Kly}_n \simeq \mathrm{H}^\bullet(\mathrm{Pet}_n)_{\mathbf{R}}$, where Pet_n is the Peterson variety, also known as the regular nilpotent Hessenberg variety for $h(i) = i + 1$.

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- Open: find a nilpotent Hessenberg-like $\mathbf{F}_q$ analogue realising $\mathrm{Kly}_{n,q}$.

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A second model for Kly_n starts from the type A_n Cartan matrix

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All real $q > 0$: the deformed fan

For $q \in \mathbf{R}_{>0}$, start from the q-Cartan matrix

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When $q = 1$, this recovers Σ_n.

Examples of the fan $\Sigma_{2,q}$

For $n = 2$,

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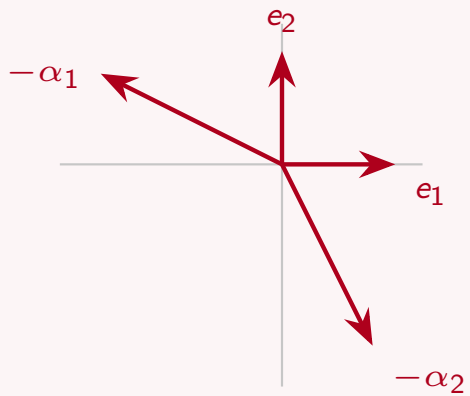
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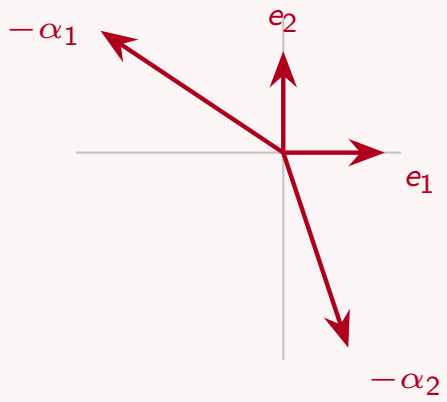


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Classical Klyachko algebra and its geometric models:

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Thank you!