DELIGNE–LUSZTIG VARIETIES, TORIC ORBIFOLDS, AND THE q-KLYACHKO ALGEBRA

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Abstract. We investigate the geometry behind the q-Klyachko algebra, introduced by Nadeau–Tewari. When q is a prime power, we show that the q-Klyachko algebra is the image of the pullback map on Chow rings $\mathrm{CH}(\mathrm{Fl}_{n+1}) \to \mathrm{CH}(\mathrm{DL}_n)$, where DL_n is a compactified Deligne–Lusztig variety inside the complete flag variety Fl_{n+1}. When q is a positive rational number, we establish a Kähler package for the q-Klyachko algebra through inputs from toric geometry.

1. Introduction

Fix $q \in \mathbf{Q}_{>0}$ and a natural number $n \in \mathbf{Z}_{>0}$. In [NT25], Nadeau and Tewari considered:

Definition 1.1. The *q-Klyachko algebra* $\mathrm{Kly}_{n,q}$ is the finite $\mathbf{Q}$-algebra

$$\mathrm{Kly}_{n,q} := \mathbf{Q}[u_1, \cdots, u_n]/\langle\, (q+1)u_i^2 = u_i u_{i+1} + q u_i u_{i-1}, \text{ for all } i = 1, \ldots, n \,\rangle,$$

with the convention $u_0 = u_{n+1} = 0$. It is equipped with a degree map $\deg_{n,q} \colon \mathrm{Kly}^n_{n,q} \to \mathbf{Q}$ by sending $u_1 \cdots u_n$ to $(n)_q! = \prod_{i=1}^{n} \frac{q^i - 1}{q-1}$.

The set of square-free monomials in u_i form a basis for $\mathrm{Kly}_{n,q}$. Thus the degree map is well-defined. The purpose of the present note is to study the geometry behind the q-Klyachko algebra.

1.1. **Motivation from probability theory.** Consider a compactly supported point measure on the set of integers of the form $\eta = \sum_{i \in \mathbf{Z}} c_i \delta_i$, where $c_i \in \mathbf{Z}_{\geq 0}$ vanishes for all but finitely many i, and δ_i is the Dirac measure at i. Fix $q_L, q_R \in \mathbf{Q}_{>0}$ with $q_L + q_R = 1$. We consider the following *random displacement* rule. Whenever $\eta(i) \geq 2$ for some i, we set

$$\eta \longmapsto \eta - \delta_i + \delta_{i-1} \text{ with probability } q_L, \text{ or } \eta - \delta_i + \delta_{i+1} \text{ with probability } q_R.$$

That is, we move 1 unit of the mass at i leftward with probability q_L and rightward with probability q_R. See Figure 1 for an illustration. A point measure with $\eta(i) \in \{0, 1\}$ for all $i \in \mathbf{Z}$ is invariant under this displacement rule.

Let $I \subset \mathbf{Z}$ be a finite subset. Denote by $p(I; \eta)$ the probability that the point measure η transforms to the point measure $\sum_{i \in I} \delta_i$ after running the random displacement process above for sufficiently many steps. Here, at every step, we select one i with $\eta(i) \geq 2$ according to some rule.

In [NT25, §5], Nadeau and Tewari showed that the structure constants for the square monomial basis of $\mathrm{Kly}_{n,q}$, known as *remixed Eulerian numbers*, compute the probability $p(I; \eta)$. More precisely, setting $q = q_L/q_R$, if η is supported on $[n]$ and

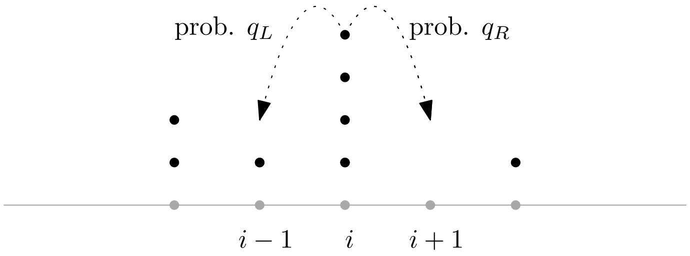

FIGURE 1. An application of the random displacement rule at i.

is such that $\sum_{1\leq i\leq n}\eta(i)=n$, then we have

$$p([n];\eta)=\frac{1}{(n)_q!}\deg_{n,q}\prod_i u_i^{\eta(i)}.$$

Moreover, this holds for any rule of selecting i to update.

1.2. **Statements of the main results.** The first geometric object related to $\mathrm{Kly}_{n,q}$ is a Deligne–Lusztig variety. Fix a finite field $k=\mathbf{F}_q$ of order q, in addition to an algebraic closure $\bar{k}$. Denote by Fl_{n+1} the complete flag variety over $\bar{k}$. Let DL_n be the compactified Deligne–Lusztig variety (§2) associated to a Coxeter element and the q-Frobenius. The space DL_n is a smooth projective subvariety of Fl_{n+1}; let $\iota\colon \mathrm{DL}_n\to\mathrm{Fl}_{n+1}$ be the evident embedding.

Theorem 1.2. *For $1\leq i\leq n$, we set L_i to be the pullback of line bundle on Fl_{n+1} associated to the ith fundamental weight for the type A_n root system. Then, we have the following isomorphism of graded $\mathbf{Q}$-algebras,*

$$\epsilon\colon \mathrm{Kly}_{n,q}\xrightarrow{\sim}\mathrm{im}(\iota^*\colon \mathrm{CH}(\mathrm{Fl}_{n+1})\to\mathrm{CH}(\mathrm{DL}_n)),\qquad u_i\longmapsto\iota^*c_1(L_i).$$

Our argument relies on a result of Langer [Lan19], in which he identified the DL_n as the de Concini–Procesi wonderful model for the arrangement $\mathrm{PG}(n,q)$ of all $\mathbf{F}_q$-hyperplanes, so $\mathrm{CH}(\mathrm{DL}_n)$ is isomorphic to the matroidal Chow ring $\mathrm{CH}(\mathrm{PG}(n,q))$ in the sense of Feichtner–Yuzvinsky [FY04]. If no confusion arises, we also denote by $\mathrm{PG}(n,q)$ the matroid associated to the arrangement.

Remark 1.3.
- (i) In [NT25, §7], Nadeau and Tewari related the q-Klyachko algebra $\mathrm{Kly}_{n,q}$ to the Schubert cycle expansion of the variety DL_n. This is an illustration of Theorem 1.2 at degree n.
- (ii) In [KK24, Remark 7.13], Katz and Kutler exhibited a ring map $\mathrm{Kly}_{n,q}\to\mathrm{CH}(\mathrm{PG}(n,q))$ and asked about the geometrisation of this map. This map is, up to change of variables, given by ϵ. Theorem 1.2, therefore, provides a positive answer to the question of Katz–Kutler.

In a different direction, we establish a second geometric interpretation for the q-Klyachko algebra, valid even when q is not a prime power.

Theorem 1.4. *For any positive rational number q, there exists a projective simplicial fan $\Sigma_{n,q}$ (§3) such that the cohomology ring $\mathrm{H}^\bullet(X_{\Sigma_{n,q}},\mathbf{Q})$ of the associated toric variety $X_{\Sigma_{n,q}}$ is isomorphic to $\mathrm{Kly}_{n,q}$ as graded $\mathbf{Q}$-algebras. Under this isomorphism, the ample cone of $X_{\Sigma_{n,q}}$ contains the cone*

$$\mathcal{K}_{n,q}\coloneqq\mathbf{R}_{>0}\,u_1+\cdots+\mathbf{R}_{>0}\,u_n\subseteq\mathrm{Kly}^1_{n,q,\mathbf{R}}.$$

This result is inspired by the work of Abe and Zeng [AZ23], who established the case when $q = 1$. Granting the theorem, we immediately deduce:

Corollary 1.5. *(i) (Kähler package) The q-Klyachko algebra $\mathrm{Kly}_{n,q}$ is a Poincaré duality algebra. Moreover, it satisfies the Hard Lefschetz theorem and Hodge-Riemann relations with respect to any class $\ell \in \mathcal{K}_{n,q}$.*

(ii) (Volume polynomial realisation) Keeping the notation and assumptions of Section 1.1. We have

$$(n)_q! \int_{X_{\Sigma_{n,q}}} (x_1u_1 + \cdots + x_nu_n)^n = \sum_{\eta} p([n];\eta) \prod_{i=1}^{n} x_i^{\eta(i)},$$

where on the left-hand side, the u_i are construed as $\mathbf{Q}$-divisors on $X_{\Sigma_{n,q}}$, and the right-hand sum runs over all point measures supported on $[n]$ with $\sum_{i=1}^n \eta(i) = n$.

The Kähler package in Item (i) is a consequence of Hodge theory for simplicial polytopes in the sense of Stanley and McMullen [Sta80, McM93]. Item (ii), in addition to the Khovanskii–Teissier inequality, implies the probability $p([n];\eta)$ satisfies log-concavity properties of the form

$$p([n];\eta)^2 \geq p([n];\eta - \delta_i + \delta_{i-1}) \cdot p([n];\eta - \delta_i + \delta_{i+1}), \quad \text{for all } i \text{ with } \eta(i) \geq 2.$$

More broadly speaking, volume polynomials are special cases of Lorentzian polynomials in the sense of Brändén and Huh [BH20, Definition 2.1, Theorem 4.6].

Finally, we say a few words about the dichotomy between the geometric situations when $q = 1$ and when q is a prime power. The arrangement $\mathrm{PG}(n,q)$ can be seen as the q-analogue of the arrangement of coordinate hyperplanes in $\mathbf{P}^n$. This comes from viewing $\mathrm{PG}(n,q)$ as the vanishing locus of the *Moore determinant*:

$$\Delta_{n,q}(x_1,\ldots,x_n) = \det \begin{pmatrix} x_0 & \cdots & x_n \\ x_0^q & \cdots & x_n^q \\ \vdots & \ddots & \vdots \\ x_0^{q^n} & \cdots & x_n^{q^n} \end{pmatrix}.$$

Note that replacing the powers $x_i^{q^j}$ by x_i^j yields the usual Vandermonde determinant, which cuts out the coordinate arrangement. Moreover, the wonderful compactification of the coordinate arrangement is the n-dimensional permutohedral variety X_{A_n}[1]. From this perspective, one can view DL_n as the q-analogue of the permutohedral variety X_{A_n}, and Theorem 1.2 as the q-analogue of the well-known isomorphism, usually attributed to Klyachko [Kly85], between $\mathrm{Kly}_{1,n}$ and the image of the restriction map $\mathrm{CH}(\mathrm{Fl}_{n+1}) \to \mathrm{CH}(X_{A_n})$. We summarise the dichotomy in the following table.

For the last row, we note that over the complex numbers, the Peterson variety Y_n has the property that its cohomology ring is the Klyachko algebra, $\mathrm{Kly}_{n,1} \simeq \mathrm{H}^\bullet(Y_n, \mathbf{Q})$. In [AZ23], Abe and Zeng established a map $Y_n \to X_{\Sigma_{1,n}}$ that realises this isomorphism in cohomology. It is natural to ask if there is a similar subvariety of Fl_{n+1} whose cohomology ring equals $\mathrm{Kly}_{n,q}$. The variety Y_n is a regular nilpotent Hessenberg variety with Hessenberg function $h = (2, 3, \ldots, n, n)$. The regular

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[1] Here, we mean the wonderful model associated to the maximal building set in the sense of [DCP95].

$q=1$	q prime power
Permutohedral toric variety X_{A_n}	Deligne–Lusztig variety DL_n
Coordinate arrangement	The arrangement $\mathrm{PG}(n,q)$ of all $\mathbf{F}_q$-hyperplanes
Klyachko algebra $\mathrm{Kly}_{n,1}$	q-Klyachko algebra $\mathrm{Kly}_{n,q}$
$\mathrm{Kly}_{n,1} \simeq \mathrm{im}(\mathrm{CH}(\mathrm{Fl}_{n+1}) \to \mathrm{CH}(X_{A_n}))$ (Klyachko [Kly85])	$\mathrm{Kly}_{n,q} \simeq \mathrm{im}(\mathrm{CH}(\mathrm{Fl}_{n+1}) \to \mathrm{CH}(\mathrm{DL}_n))$ (Theorem 1.2)
The fan $Y(A_n) = \Sigma_{1,n}$ in [AZ23, Blu15, HMSS24]	The fan $\Sigma_{n,q}$ constructed in §3.
$\mathrm{Kly}_{n,1} \simeq \mathrm{H}^\bullet(X_{\Sigma_{1,n}}, \mathbf{Q})$ (Abe–Zeng [AZ23])	$\mathrm{Kly}_{n,q} \simeq \mathrm{H}^\bullet(X_{\Sigma_{n,q}}, \mathbf{Q})$ (Theorem 1.4)
Peterson variety Y_n	???

semisimple Hessenberg variety associated to the same Hessenberg function h is exactly the permutohedral variety X_{A_n}, whose role is taken by DL_n in our q-analogue situation.

Concerning organisation, we start in Section 2 explaining the connection among the q-Klyachko algebra, the Deligne–Lusztig variety, and the projective geometry matroid, proving Theorem 1.2. In §3, we give the construction of the fan $\Sigma_{n,q}$ and prove Theorem 1.4.

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2. The Deligne–Lusztig variety of a Coxeter element

Let us first introduce some notation. First, we define the q-integers and q-factorials

$$(m)_q := \frac{1-q^m}{1-q} = 1 + \cdots + q^{m-1}, \quad (m)_q! := \prod_{1\le i\le m} (i)_q.$$

Put $k = \mathbf{F}_q$ for q a prime power. Let $G = \mathrm{GL}_{n+1}$ be the general linear group over k and $B \subseteq G_{\bar{k}}$ a Borel subgroup. We define $\mathrm{Fl}_{n+1} := G_{\bar{k}}/B$ as the (complete) flag variety over $\bar{k}$. Let $\pi_d \colon \mathrm{Fl}_{n+1} \to \mathrm{Gr}(d, n+1)$ be the map that forgets all but the dth subspace.

Definition 2.1. Let $w \in \mathfrak{S}_{n+1}$. The *Deligne–Lusztig variety* $\mathrm{DL}_n^\circ(w)$ associated to w is defined as the fibre product

$$\Gamma_{\mathrm{Frob}} \times_{\mathrm{Fl}_{n+1}\times\mathrm{Fl}_{n+1}} [G.(B, wB)]$$

where $\mathrm{Frob}\colon \mathrm{Fl}_{n+1} \to \mathrm{Fl}_{n+1}$ is the absolute Frobenius, G acts diagonally on $\mathrm{Fl}_{n+1} \times \mathrm{Fl}_{n+1}$, and $G.(B, wB)$ is the G-orbit of (B, wB). Now, consider the permutation $c = (1,2)(2,3)\cdots(n,n+1) \in \mathfrak{S}_{n+1}$. The *Coxeter variety* is defined to be $\mathrm{DL}_n^\circ := \mathrm{DL}_n^\circ(c)$. The *compactified Coxeter variety* DL_n is closure of DL_n° in Fl_{n+1}.

The space DL_n is a smooth projective variety of dimension n with simple normal crossing boundary $\mathrm{DL}_n \smallsetminus \mathrm{DL}_n^\circ$. Analysing the fibre product above, one can describe DL_n° very explicitly:

Observation 2.2. Restricting the map $\pi_1\colon \mathrm{Fl}_{n+1} \to \mathbf{P}^n$ yields an isomorphism:

$$\begin{aligned}\pi_1|_{\mathrm{DL}_n^\circ}\colon \mathrm{DL}_n^\circ \xrightarrow{\sim} &\{P \in \mathbf{P}^n \mid P, \ldots, \mathrm{Frob}^n P \text{ are in general position}\}\\ &= \mathbf{P}^n \smallsetminus \bigcup\{H : H \text{ is a } k\text{-rational hyperplane}\}, \quad (V_i) \longmapsto V_1.\end{aligned}$$

The target is the complement of the hyperplane arrangement $\mathrm{PG}(n, q)$, the arrangement of all k-rational hyperplanes in $\mathbf{P}^n$.

Let DL_n' be the wonderful compactification of the arrangement complement DL_n° in the sense of de Concini and Procesi [DCP95]. This is obtained as a series of blow-ups on $\mathbf{P}^n$: first blow up all k-points, then the strict transforms of all k-lines, and then k-planes, and so on. Let $\pi'\colon \mathrm{DL}_n' \to \mathbf{P}^n$ be the composition of these blow-ups.

Langer observed the following; see [Lan19, Proposition 7.1],

Proposition 2.3 (Langer)**.** *We have the following equality,*

$$\begin{tikzcd} \mathrm{DL} \arrow[r, "\pi_1"] \arrow[d, "="] & \mathbf{P}^n \arrow[d, "="] \\ \mathrm{DL}' \arrow[r, "\pi'"] & \mathbf{P}^n \end{tikzcd}$$

In particular, $\mathrm{DL}_n = \mathrm{DL}_n'$ *on the nose.*

Corollary 2.4. *The rational Chow ring of* DL_n *is given by the Chow ring of the matroid* $\mathrm{PG}(n, q)$*:*

$$\begin{aligned}\mathrm{CH}(\mathrm{DL}_n)_{\mathbf{Q}} &\simeq \mathrm{CH}(\mathrm{PG}(n,q))_{\mathbf{Q}}\\ &:= \frac{\mathbf{Q}[x_F : F \subset \mathbf{P}^n \textit{ a } k\textit{-rational proper subspace}]}{I_{n,q} + J_{n,q}},\end{aligned}$$

where $I_{n,q}$ *is generated by linear relations*

$$\sum_{F \subset H} x_F - \sum_{F \subset H'} x_F, \quad \textit{for any pair of } k\textit{-rational hyperplanes } H, H' \subset \mathbf{P}^n,$$

and $J_{n,q}$ *is generated by monomials*

$$x_F x_{F'} \quad \textit{for every pair } F, F' \textit{ of subspaces not contained in one another.}$$

Proof. Thanks to Proposition 2.3, we can compute the Chow ring of DL_n by viewing it as a wonderful model of hyperplane arrangements. Then the assertion follows immediately from the presentation of the Chow ring of wonderful models established in [FY04, Corollary 2]. □

Here, the class x_F comes from the exceptional divisor of blowing up (the strict transform of) a k-linear subspace $F \subset \mathbf{P}^n$. Picking a k-rational hyperplane $H \subseteq \mathbf{P}^n$, we define $\alpha := \sum_{F \subset H} x_F$—the relations from the ideal $I_{n,q}$ guarantee the linear equivalence class of α is independent of the choice of the hyperplane H. Geometrically, this class of α is the first Chern class $c_1(\pi_1^* \mathcal{O}_{\mathbf{P}^n}(1))$.

Recalling that $N = (n+1)_q - 1$, we define divisor classes γ_k on X_{A_N} as the pullback of the class of the N-dimensional hypersimplex $\Delta(N+1, (k)_q)$ in the sense of [BST23] and [KK24]; geometrically, this comes from pulling back the Plücker line bundle on the Grassmannian $\mathrm{Gr}((k)_q, N+1)$ via the composite

$$X_{A_N} \hookrightarrow \mathrm{Fl}_{N+1} \to \mathrm{Gr}((k)_q, N+1).$$

We will also denote by $\gamma_k \in \mathrm{CH}^1(\mathrm{DL}_n)$ the restriction of γ_k to DL_n. Also, since DL_n is a subvariety of Fl_{n+1}, we can define $L_k \in \mathrm{CH}^1(\mathrm{DL}_n)$ as the restriction of the first Chern class of the kth Plücker line bundle on Fl_{n+1}. In the case when $q = 1$ and DL_n is replaced by the permutohedral variety X_{A_n}, the two types of divisors agree; $L_i = \gamma_i$ for all $1 \leq i \leq n$. The following lemma is a q-deformed version of this coincidence.

Lemma 2.5. *For all $1 \leq i \leq n$, we have $\gamma_i = q^i L_i$ as divisor classes on DL_n.*

Proof. By [Lan19, Lemma 6.2], we have

$$L_i = (n+1-i)_q \alpha - \sum_{F:\ \operatorname{codim} F \geq i} (\operatorname{codim} F - i)_q x_F.$$

On the other hand, by [KK24, Lemma 2.4], we have

$$\gamma_i = ((n+1)_q - (i)_q)\alpha - \sum_{F:\ \operatorname{codim} F \geq i} ((\operatorname{codim} F)_q - (i)_q) x_F.$$

The claim follows immediately from the identity $\frac{(a)_q - (b)_q}{(a-b)_q} = q^b$. □

Finally, the matroid $\mathrm{PG}(n, q)$ is the special case of a more general class of matroids called *perfect matroid designs*. In [KK24, Lemma 7.10], the authors established quadratic relations among γ_i for general perfect matroid designs. Applying their formula to $\mathrm{PG}(n, q)$, we obtain the following.

Lemma 2.6. *The divisor classes γ_i satisfy the q-Klyachko relation*

$$(q+1)\gamma_i^2 = \gamma_i\gamma_{i+1} + q\gamma_i\gamma_{i-1}, \quad \textit{for all } 1 \leq i \leq n.$$

We are now in a position to establish the main claim.

Proof of Theorem 1.2. First, we check that Plücker classes L_i also satisfy a q-Klyachko relation, we have

$$\begin{aligned}(q+1)L_i^2 &= \frac{q+1}{q^{2i}}\gamma_i^2 = \frac{1}{q^{2i}}(\gamma_i\gamma_{i+1} + q\gamma_i\gamma_{i-1}) \\ &= \frac{1}{q^{2i}}(q^{2i+1}L_iL_{i+1} + q^{2i}L_iL_{i-1}) = qL_iL_{i+1} + L_iL_{i-1},\end{aligned}$$

by Lemma 2.5. Therefore, the map $\iota^* \colon \mathrm{CH}(\mathrm{Fl}_{n+1}) \to \mathrm{CH}(\mathrm{DL}_n)$ factors through the map $\tau \colon \mathrm{Kly}_{n,q} \to \mathrm{CH}(\mathrm{DL}_n)$ that sends u_i to L_{n-i}. Since the Chow ring $\mathrm{CH}(\mathrm{Fl}_{n+1})$ is generated by L_i, we get a surjection

$$\tau \colon \mathrm{Kly}_{n,q} \twoheadrightarrow \operatorname{im} \iota^*.$$

But [KK24, Theorem 7.12] combined with Lemma 2.5 shows the degree map on $\mathrm{CH}(\mathrm{DL}_n)$ restricts to the degree map on $\mathrm{Kly}_{n,q}$ given by q-divided symmetrisation, so τ must also inject. □

3. THE TORIC ORBIFOLD $X_{\Sigma_{n,q}}$

3.1. **A q-deformation of the type A Cartan matrix.** We give a different construction that realises the q-Klyachko algebra. The plan is to exhibit the q-Klyachko as the Stanley–Reisner ring of a toric orbifold, and then the Kähler package follows from Hodge theory for simple polytopes.

Assume now $q \in \mathbf{Q}_{>0}$. Consider the n-by-n matrix,

$$A(n,q) = \begin{pmatrix} q+1 & -1 & & \\ -q & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -q & q+1 \end{pmatrix}$$

where the omitted entries are zero. When $q = 1$, this is the Cartan matrix of the root system of type A_n. This matrix also appeared in [NT25, Lemma 3.6]. The following list of properties is clear:

Lemma 3.1. *(i) The matrix $A(n,q)$ has determinant $(n+1)_q$.*

(ii) For any subset $J \subseteq [n]$, the submatrix $A(n,q)_{i,j\in J}$ is nonsingular.

(iii) The matrix $A(n,q)$ satisfies $(A(n,q)^{-1})_{ij} > 0$ for all $1 \leq i,j \leq n$.

(iv) For any subset $J \subseteq [n]$, the submatrix $A_J = A(n,q)_{i,j\in J}$ satisfies $(A_J^{-1})_{ij} \geq 0$ for all $i,j \in J$.

Recall that when $q = 1$, the matrix $A(n,1)_{i,\,j\in J}$ is a Cartan matrix of the root system of type A_n for any $J \subseteq [n]$. Then Item (iv) above is an illustration of the fact that the inverse $[A(n,1)_{i,\,j\in J}]^{-1}$ consists of nonnegative rational numbers [LT92].

3.2. **The simplicial fan $\Sigma_{n,q}$: construction and properties.** Denote by e_i the standard basis vector in $N = \mathbf{Z}^n$, and let $\alpha_1, \ldots, \alpha_n \in \mathbf{Q}^n$ be the column vectors of $A(n,q)$.

Let $\Sigma_{n,q}$ be the set of cones of the form

$$\sigma_{J,K} = \operatorname{cone}\{e_i : i \in J\} - \operatorname{cone}\{\alpha_k : k \in K\}$$

for disjoint subsets $J, K \subsetneq [n]$. Here, our convention is that $\operatorname{cone}\varnothing$ is the origin $\{0\}$, so that $\sigma_{\varnothing,\varnothing} = \{0\}$. As a matter of notation, we write $A = A(n,q)$ and $A_J = A_{i,j\in J}$.

Using Lemma 3.1, one can easily observe:

Lemma 3.2. *For $J, K \in [n]$ so that $J \cap K = \varnothing$, the cone $\sigma_{J,K}$ is a strongly convex rational polyhedral cone in $N_{\mathbf{R}}$ of dimension $\dim \sigma_{J,K} = \#J + \#K$. Moreover, given $\sigma_{J,K}, \sigma_{P,Q} \in \Sigma_{n,q}$, we have $\sigma_{J,K} \cap \sigma_{P,Q} = \sigma_{J\cap P, K\cap Q}$.*

Corollary 3.3. *The set $\Sigma_{n,q}$ is a simplicial fan. In other words, the toric variety $X_{\Sigma_{n,q}}$ associated to the fan $\Sigma_{n,q}$ is $\mathbf{Q}$-factorial.*

Proof. Firstly, we check that $\Sigma_{n,q}$ is a fan. Indeed, given a cone $\sigma_{J,K} \in \Sigma_{n,q}$, every nonempty face is given by the cone $\sigma_{J',K'} \in \Sigma_{n,q}$ for $J' \subseteq J$ and $K' \subseteq K$. Secondly, by the second assertion of Lemma 3.2, the intersection of any two cones $\sigma_{J,K}$ and $\sigma_{P,Q}$ in $\Sigma_{n,q}$ is a face of both, namely, $\sigma_{J,K} \cap \sigma_{P,Q}$. Finally, by the first assertion of Lemma 3.2, the fan $\Sigma_{n,q}$ is simplicial. □

Next, we show the toric variety $X_{\Sigma_{n,q}}$ is projective. The argument goes by first establishing completeness and then detecting amplitude *via* the toric version of Kleiman's criterion.

Lemma 3.4. *The fan $\Sigma_{n,q}$ is complete.*

Proof. The proof here is the same as the one in [AZ23, Proposition 4.3]. The proof in *loc.cit.* goes through upon replacing references to [AZ23, Lemma 3.3 and Lemma 3.5] by the first and second assertion of Lemma 3.2. □

Considering the orbit-cone correspondence for toric varieties [CLS11, §3.2], we denote by $D_{-\alpha_i}$ the torus invariant Weil divisor on $X_{\Sigma_{n,q}}$ corresponding to the ray $\sigma_{\varnothing,i} = \operatorname{cone}(-\alpha_i)$ for $i \in [n]$. Similarly, denote by D_{e_k} the torus invariant Weil divisor that corresponds to the ray $\sigma_{k,\varnothing} = \operatorname{cone} e_k$.

Proposition 3.5. *For a torus invariant irreducible curve $C \subseteq X_{\Sigma_{n,q}}$, and an invariant Weil divisor $D_{-\alpha_i}$ that corresponds to the ray generator $-\alpha_i$, the intersection number $(D_{-\alpha_i} \cdot C)$ is nonnegative. Moreover, given an invariant curve C, there exists $\ell \in [n]$ such that the corresponding Weil divisor $D_{-\alpha_\ell}$ satisfies $(D_{-\alpha_\ell} \cdot C) > 0$.*

Proof. Under the orbit-cone correspondence, a torus invariant irreducible curve C corresponds to a codimension-one cone of the form $\sigma_{J,K}$ with $J \cup K = [n] \smallsetminus \ell$ for some ℓ. The cone $\sigma_{J,K}$ is contained in the maximal cones $\sigma_{J\cup\ell,K}$ and $\sigma_{J,K\cup\ell}$. By [CLS11, Proposition 6.4.4], there exists a unique up to homothety wall relation among the $n+1$ ray generators in $\sigma_{J\cup\ell,K}$ and $\sigma_{J,K\cup\ell}$, given by the linear relation

$$-(D_{-\alpha_\ell} \cdot C)\alpha_\ell - \sum_{j\in J}(D_{-\alpha_j} \cdot C)\alpha_j + \sum_{k\in K}(D_{e_k} \cdot C)e_k + (D_{e_\ell} \cdot C)e_\ell = 0; \tag{3.2.1}$$

moreover, the intersection numbers $(D_{-\alpha_\ell} \cdot C)$ and $(D_{e_\ell} \cdot C)$ are strictly positive—this proves the second assertion. Put $J' = J \cup \ell$. Pairing Eq. (3.2.1) against e_i for $i \in J'$, we obtain a system of linear equations

$$\sum_{j\in J'} A_{ij}(D_{-\alpha_j} \cdot C) = \delta_{i\ell}\,(D_{e_i} \cdot C) \text{ for all } i \in J'.$$

By Lemma 3.1, the submatrix $A_{J'}$ is nonsingular, and its inverse $A_{J'}^{-1}$ has nonnegative entries. Solving for the linear equation and noting that $(D_{e_i} \cdot C) > 0$, we have $(D_{-\alpha_j} \cdot C) \geq 0$ for all $j \in J'$. This yields the first assertion. ☐

By Lemma 3.4, the toric variety $X_{\Sigma_{n,q}}$ is complete. Recall that the toric Kleiman criterion asserts that if D is a Cartier divisor on a complete toric variety X, then D is ample if and only if $(D \cdot C) > 0$ for all torus invariant irreducible curves C. Combining this with Proposition 3.5 and the fact that $X_{\Sigma_{n,q}}$ is $\mathbf{Q}$-factorial (Corollary 3.3), one immediately deduces

Corollary 3.6. *Given positive rational numbers $a_i \in \mathbf{Q}_{>0}$ for $1 \leq i \leq n$, the $\mathbf{Q}$-divisor $D = \sum_{i=1}^n a_i D_{-\alpha_i}$ is $\mathbf{Q}$-ample. That is, a sufficiently large multiple of D is Cartier and ample.*

Example 3.7 (Case $n = q = 2$)**.** In Figure 2, we visualise the fan $\Sigma_{2,2}$ in $N_{\mathrm{R}} \simeq \mathbf{R}^2$. The 4 maximal cones of $\Sigma_{2,2}$ are the labelled regions. The standard basis vectors e_i generate the cone $\sigma_{i,\varnothing}$, for $i = 1, 2$. The remaining two vectors are $-\alpha_1 = -3e_1+2e_2$ and $-\alpha_2 = e_1 - 3e_2$, which gives the cones $\sigma_{\varnothing,1}$ and $\sigma_{\varnothing,2}$ respectively. One observes that $\Sigma_{2,2}$ is the normal fan of a lattice polytope that is combinatorially equivalent to a 2-dimensional cube.

3.3. Cohomology of the toric orbifold $X_{\Sigma_{n,q}}$.

This subsection is devoted to proving Theorem 1.4. Since the toric variety $X_{\Sigma_{n,q}}$ is $\mathbf{Q}$-factorial and projective, the rational Betti cohomology ring $\mathrm{H}^\bullet(X_{\Sigma_{n,q}}, \mathbf{Q})$ can be computed by Stanley–Reisner theory [CLS11, Chap. 12].

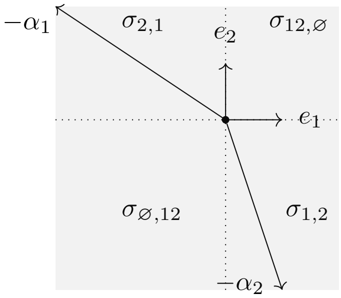

FIGURE 2. Illustration of the fan $\Sigma_{2,2}$, maximal cones labelled by two-part partitions of $[2]$.

Proof of Theorem 1.4. By [CLS11, Section 12.4], we have the ring isomorphism

$$\begin{aligned} \mathrm{SR}\colon \mathbf{Q}[X_i, Y_i : i \in [n]]/(\mathcal{I}_{n,q} + \mathcal{J}_{n,q}) &\to \mathrm{H}^\bullet(X_{\Sigma_{n,q}}, \mathbf{Q}); \\ X_i \longmapsto [D_{-\alpha_i}], \quad Y_i \longmapsto [D_{e_i}], &\quad \text{for all } i \in [n], \end{aligned}$$

where the ideal $\mathcal{I}_{n,q}$ is generated by quadrics X_iY_i for all $i \in [n]$, and the ideal $\mathcal{J}_{n,q}$ is generated by linear forms $-\sum_{1\le j\le n} A(n,q)_{ij}X_j + Y_i$, for all $i \in [n]$. By our definition of $A(n,q)$, the ring map SR factors through an isomorphism

$$\mathrm{SR}'\colon \mathbf{Q}[X_1, \dots, X_n]/\mathcal{I}'_{n,q} \to \mathrm{H}^\bullet(X_{\Sigma_{n,q}}, \mathbf{Q}), \quad X_i \longmapsto [D_{-\alpha_i}],$$

where the ideal $\mathcal{I}'_{n,q} \subseteq \mathbf{Q}[X_1, \dots, X_n]$ is generated by quadrics

$$X_i\left((q+1)X_i - X_{i+1} - qX_{i-1}\right), \quad \text{for } 1 \le i \le n,$$

where we set $X_0 = X_{n+1} = 0$. But the q-Klyachko algebra $\mathrm{Kly}_{n,q}$ is isomorphic to the source $\mathbf{Q}[X_1, \dots, X_n]/\mathcal{I}'_{n,q}$ of SR' by sending u_i to X_i. The second assertion of the theorem follows from Corollary 3.6. ☐

3.4. A Kähler package for the q-Klyachko algebra.

In this subsection, we use the Hodge theory for simplicial polytopes to prove Corollary 1.5.

Definition 3.8. Let $A^\bullet = \bigoplus_{i=0}^d A^i$ be a graded Artinian $\mathbf{Q}$-algebra of finite type, in addition to an isomorphism $\deg\colon A^d \to \mathbf{Q}$. We say that $(A^\bullet, \deg)$ *satisfies the Kähler package* with respect to an element $\ell \in A^1$ if the following conditions are satisfied,

(i) (Poincaré duality) For $0 \le k \le d/2$, the bilinear form

$$A^k \times A^{d-k} \to \mathbf{Q}, \quad (\eta_1, \eta_2) \longmapsto \deg \eta_1\eta_2$$

is a perfect pairing.

(ii) (Hard Lefschetz theorem) For every integer $0 \le k \le d/2$, the multiplication map

$$\times \ell^{d-2k}\colon A^k \to A^{d-k}, \quad \eta \longmapsto \ell^{d-2k}\eta$$

is an isomorphism.

(iii) (Hodge index theorem) For every integer $0 \le k \le d/2$, the bilinear form

$$A^k \times A^k \to \mathbf{Q}, \quad (\eta_1, \eta_2) \longmapsto (-1)^k \deg(\ell^{d-2k}\eta_1\eta_2)$$

is positive definite upon restricting to the kernel of multiplication by ℓ^{d-2k+1}.

Proof of Corollary 1.5. For a simplicial lattice polytope P with normal fan Σ_P, Stanley and McMullen [Sta80, McM93] showed that on the toric variety X_{Σ_P} associated to the fan Σ_P, the rational Betti cohomology ring $\mathrm{H}^\bullet(X_{\Sigma_P}, \mathbf{Q})$ satisfies Poincaré duality and the hard Lefschetz theorem with respect to any ample divisor class $\ell \in \mathrm{H}^2(X_{\Sigma_P}, \mathbf{Q})$. Moreover, in *loc.cit.*, McMullen also established the Hodge index theorem for any ample class ℓ. Since the fan $\Sigma_{n,q}$ is projective and simplicial, Item (i) follows. Concerning Item (ii), by [CLS11, Lemma 12.5.2], we have $\int_{X_{\Sigma_{n,q}}} D_{-\alpha_1} \cdots D_{-\alpha_n} = |\det A(n,q)|^{-1} = (n)_q!^{-1}$. Therefore we have

$$\int_{X_{n,q}} \prod_i D_{-\alpha_i}^{\eta(i)} = \frac{1}{[(n)_q!]^2} \deg_{n,q} \prod_i u_i^{\eta(i)} = \frac{p([n];\eta)}{(n)_q!}$$

for any $\eta\colon [n] \to \mathbf{Z}_{\geq 0}$ with $\sum_i \eta(i) = n$. ☐

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